# MAJORANTS AND $L^p$ NORMS†

BY

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ABSTRACT

It is shown that if $G$ is an infinite compact abelian group, then $L^p(G)$ has the upper majorant property only if $p$ is even or $p = \infty$.

## 1. Introduction

Let $G$ be a compact, abelian group with dual group $\Gamma$. A trigonometric polynomial $g$ is said to *majorize* another trigonometric polynomial $f$ if $\hat{g}(\gamma) \geqq |\hat{f}(\gamma)|$ for all $\gamma$ in $\Gamma$. The space $L^p(G)$ is said to have the *upper majorant property* if there exists a constant $A(p)$ such that

$$\|f\|_p \leqq A(p)\|g\|_p$$

whenever $g$ majorizes $f$. It is easy to see for even integers $p$ and for $p = \infty$, that $L^p(G)$ has the upper majorant property with constant $A(p) = 1$. The aim of this paper is to show that if the group $G$ is infinite, and if the index $p$ is finite and not an even integer, then $L^p(G)$ does *not* have the upper majorant property.

Before giving the proof, we discuss the history of the problem. Hardy and Littlewood [3, p. 305] gave an example, for the circle group $T$, of trigonometric polynomials $f$ and $g$ such that $g$ majorizes $f$ but $\|f\|_3 > \|g\|_3$; thus, if $L^3(T)$ has the upper majorant property, the constant $A(3)$ must be strictly greater than 1. Boas [2, p. 255] used the same method to show that if $2 < p < \infty$ and $p$ is not an even integer, then $L^p(T)$ cannot have the upper majorant property with unit constant; he also showed that if $1 \leqq p < 2$, then $L^p(T)$ does not have the upper majorant property. Finally, Bachelis [1, p. 121] used a suggestion of Katznelson

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to prove that $L^p(T)$ has the upper majorant property only if it has the property with $A(p) = 1$; hence $L^p(T)$ does not have the upper majorant property if $2 < p < \infty$ and $p$ is not an even integer. Bachelis also showed that if $G$ is infinite, and $1 \leqq p < 2$, then $L^p(G)$ does not have the upper majorant property.

The main result of the present paper is not surprising, and in most cases the proof is like the one for the circle group. That is, we show that if $L^p(G)$ has the upper majorant property, then it has the property with unit constant; then we exhibit trigonometric polynomials $f$ and $g$, analogous to those of Hardy and Littlewood, such that $g$ majorizes $f$ and $\|f\|_p > \|g\|_p$. Surprisingly, the analogy fails in one special case, namely when $G$ is an infinite product of groups of order 2 and $p$ is an odd integer. Fortunately, there is a different family of trigonometric polynomials $f$ and $g$ that can be shown to have the desired properties in this special case.

## 2. Proof of the main result

THEOREM. *Suppose that $G$ is an infinite compact abelian group, that $0 < p < \infty$, and that $p$ is not an even integer. Then $L^p(G)$ does not have the upper majorant property.*

PROOF. Without loss of generality, suppose that the Haar measure of $G$ is 1. Fix an index $p$ satisfying the above conditions. Since the theorem is already known for $1 \leqq p < 2$, the case where $p > 2$ is the most interesting, but no such restriction on $p$ will be needed in the proof.

The following fact is implicit in [1, p. 121]. It allows us to reduce the proof of the theorem to a number of special cases.

LEMMA. *Suppose that $G_1$ is a compact abelian group and that $L^p(G_1)$ does not have the upper majorant property. Suppose that the dual group of $G_1$ is isomorphic to a subgroup of the dual group of $G$. Then $L^p(G)$ does not have the upper majorant property.*

PROOF OF THE LEMMA. Denote the dual group of $G$ by $\Gamma$ and the dual group of $G_1$ by $\Gamma_1$. By assumption, there exists an injective homomorphism $\alpha : \Gamma_1 \to \Gamma$. The dual homomorphism $\alpha^* : G \to G_1$ is continuous and surjective. In other words, we are assuming that $G_1$ is a quotient group of $G$. Normalize the Haar measure on $G_1$ so that $G_1$ has mass 1.

Given $A > 0$, choose trigonometric polynomials $f_1$ and $g_1$ on $G_1$ so that $g_1$ majorizes $f_1$ and

$$\|f_1\|_p > A\|g_1\|_p.$$

Define functions $f$ and $g$ on $G$ by $f(x) = f_1(\alpha^*(x))$ and $g(x) = g_1(\alpha^*(x))$. It is easy to verify that these functions are trigonometric polynomials, that $g$ majorizes $f$, and that $\|f\|_p = \|f_1\|_p$ and $\|g\|_p = \|g_1\|_p$. Hence

$$\|f\|_p > A\|g\|_p,$$

and it follows that $L^p(G)$ does not have the upper majorant property.

Returning to the proof of the theorem, we note that since $G$ is infinite, so is its dual group $\Gamma$. We consider 5 possibilities.

*Case* 1 [1, p. 121]. *The group* $\Gamma$ *has an element of infinite order*

Recall that $L^p(T)$ does not have the upper majorant property. This fact is usually only stated for $p \geqq 1$, but the methods of [1, p. 121] and [2, p. 255] work equally well for $p < 1$. The dual group of $T$ is the group $Z$ of integers. By assumption, a copy of $Z$ is contained in $\Gamma$; by the lemma, $L^p(G)$ does not have the upper majorant property.

*Case* 2. *Every element of* $\Gamma$ *has finite order but there exist elements of arbitrarily large, finite order*

We proceed as in the proof of the lemma. Given $A > 0$, choose trigonometric polynomials $f_1$ and $g_1$ on $T$ so that $g_1$ majorizes $f_1$ and

$$\|f_1\|_p > A\|g_1\|_p.$$

Let $\gamma$ be an element of $\Gamma$ having order $r$. The image $\gamma(G)$ is the set $\{\omega_j\}_{j=1}^r$ of all $r$th roots of unity, and for each $j$ the set of $x$ in $G$ for which $\gamma(x) = \omega_j$ has measure $1/r$. Let $f(x) = f_1(\gamma(x))$ and $g(x) = g_1(\gamma(x))$. Then $f$ and $g$ are trigonometric polynomials on $G$, and $g$ majorizes $f$.

Now

$$\int_G |f(x)|^p dx = \frac{1}{r}\sum_{j=1}^{r} |f_1(\omega_j)|^p.$$

The right side of this equation is a Riemann sum for

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_1(e^{i\theta})|^p d\theta \equiv \|f_1\|_p^p.$$

Similarly, $\|g\|_p^p$ is equal to a Riemann sum for $\|g_1\|_p^p$. Therefore, if $r$ is sufficiently large,

$$\|f\|_p > A\|g\|_p.$$

Hence $L^p(G)$ does not have the upper majorant property.

In the remaining cases, there is a bound $B < \infty$ on the orders of the elements of $\Gamma$. It follows [4, p. 255] that $\Gamma$ is a direct sum $\sum_\alpha Z(r_\alpha)$ of cyclic groups of order $r_\alpha \leqq B$. Since $\Gamma$ is infinite, there is an integer $r > 1$ such that $r_\alpha = r$ for infinitely many $\alpha$. Thus $\Gamma$ contains a subgroup isomorphic to the direct sum $\sum_{j=1}^\infty Z(r)_j$ of countably many copies of $Z(r)$. In view of the lemma, it suffices to assume that

$$\Gamma = \sum_{j=1}^{\infty} Z(r)_j;$$

then $G$ is the complete direct sum $\prod_{j=1}^\infty Z(r)_j$.

We first establish that if $L^p(G)$ has the upper majorant property, then it has the property with unit constant. Indeed, suppose to the contrary that there exist polynomials $f$ and $g$, with $g$ majorizing $f$, such that

$$\|f\|_p > A\|g\|_p,$$

for some constant $A > 1$. View each summand $Z(r)_j$ as a subgroup of $\Gamma$ and let $\gamma_j$ be a generator of $Z(r)_j$. Then for some $n$ we have that $f$ and $g$ are linear combinations of products of powers of $\gamma_1, \gamma_2, \cdots, \gamma_n$. Form the corresponding combinations of products of powers of $\gamma_{n+1}, \gamma_{n+2}, \cdots, \gamma_{2n}$, and denote the new polynomials by $f_1$ and $g_1$. The product polynomial $ff_1$ is majorized by $gg_1$. Because $f$ and $f_1$ are independent, we have that

$$\|ff_1\|_p = \|f\|_p\|f_1\|_p = \|f\|_p^2.$$

Similarly, $\|gg_1\|_p = \|g\|_p^2$. Therefore

$$\|ff_1\|_p > A^2\|gg_1\|_p.$$

Iteration of this process shows that $L^p(G)$ does not have the upper majorant property.

*Case* 3. *The order $r$ is strictly greater than* 2

The polynomials that we consider are analogues of those introduced for the circle group by Hardy and Littlewood [3, p. 305]. For small positive $t$, and an integer $n$ to be chosen later, let

$$u(x) = t[\gamma_1(x) + \cdots + \gamma_n(x)]$$

and

$$v(x) = t^n\gamma_1(x)\gamma_2(x)\cdots\gamma_n(x).$$

Then let

$$f = 1 + u - v \text{ and } g = 1 + u + v.$$

Clearly, $g$ majorizes $f$.

Choose $t$ small enough that both $f$ and $g$ take all their values in the open disc of radius 1 centred at 1. Then, using the branch of $z^{p/2}$ that is analytic in this disc and that takes the value 1 at $z = 1$, consider the functions $f^{p/2}$ and $g^{p/2}$. These are actually trigonometric polynomials on $G$ because they depend only on the values of $\gamma_1, \gamma_2, \cdots, \gamma_n$ and must therefore be linear combinations of products of powers of these characters. By the binomial theorem,

$$g^{p/2} = \sum_{s=0}^{\infty} \binom{p/2}{s} (u + v)^s.$$

Expand each term $(u + v)^s$ by the binomial theorem; for small $t$ the resulting double series for $g^{p/2}$ converges absolutely, and all rearrangements also converge to $g^{p/2}$. Let $h$ be the sum of all the terms in the double series that involve even powers of $v$ and let $k$ be the sum of all the terms that involve odd powers of $v$. Then

$$g^{p/2} = h + k \quad \text{and} \quad f^{p/2} = h - k.$$

Like $f^{p/2}$ and $g^{p/2}$, the functions $h$ and $k$ are just real-linear combinations of products of powers of $\gamma_1, \gamma_2, \cdots, \gamma_n$. That is, $\hat{h}$ and $\hat{k}$ are real-valued and supported by the subgroup of $\Gamma$ generated by $\gamma_1, \gamma_2, \cdots, \gamma_n$; denote this subgroup by $\Gamma_1$.

Now

$$\begin{aligned} \| g \|_p^p - \| f \|_p^p &= \| g^{p/2} \|_2^2 - \| f^{p/2} \|_2^2 \\ &= 2 \int_G [h(x)\bar{k}(x) + \bar{h}(x)k(x)]dx \\ &= 4 \sum_{\gamma \in \Gamma_1} \hat{h}(\gamma)\hat{k}(\gamma). \end{aligned}$$

We consider the behaviour of the terms in this sum as $t \to 0$.

The elements of $\Gamma_1$ have the form $\gamma = \prod_{j=1}^{n} \gamma_j^{m_j}$, with $0 \leqq m_j < r$ for all $j$. Suppose first that all the $m_j$ are equal to 1. The multinomial expansion of $u^n$ contains the term $t^n n! \gamma$. All other occurrences of $\gamma$ in the expansion of $h$ are accompanied by the factor $t^{n+r}$, or by some higher power of $t$. Thus, as $t \to 0$,

$$\hat{h}(\gamma) = \binom{p/2}{n} n!\, t^n + o(t^n).$$

Similarly,

$$\hat{k}(\gamma) = \frac{p}{2} t^n + o(t^n).$$

Hence

$$\hat{h}(\gamma)\hat{k}(\gamma) = \binom{p/2}{n} n!\, (p/2)t^{2n} + o(t^{2n}).$$

Next suppose that $\gamma = \prod_{j=1}^{n}\gamma_j^{m_j}$ with all $m_j \geqq 1$ and some $m_j > 1$. Let $m = \sum_{j=1}^{n} m_j$; then $m > n$. As above, we have that, as $t \to 0$,

$$\hat{h}(\gamma) = O(t^m) \text{ and } \hat{k}(\gamma) = O(t^m).$$

Hence

$$\hat{h}(\gamma)\hat{k}(\gamma) = O(t^{2m}) = o(t^{2n}).$$

Finally, suppose that some $m_j = 0$; let $l$ be the number of $m_j$s that are 0. Then $m \geqq n - l$, and

$$\hat{h}(\gamma) = O(t^m) = O(t^{n-l}).$$

On the other hand, to obtain the term $\gamma$ in the expansion of $k$, one must multiply $v$ by $\prod_{(m_j=0)}\gamma_j^{r-1}$ and by the accompanying powers of $t$. It follows that

$$\hat{k}(\gamma) = O(t^n t^{l(r-1)}) = O(t^{n+2l}),$$

because $r > 2$. Therefore

$$\hat{h}(\gamma)\hat{k}(\gamma) = O(t^{2n+l}) = o(t^{2n}).$$

Combining these estimates we have that, as $t \to 0$,

$$\| g \|_p^p - \| f \|_p^p = \binom{p/2}{n} n!\ (2p)t^{2n} + o(t^{2n}).$$

Since $p$ is not an even integer, we can choose $n$ so that $\binom{p/2}{n} < 0$. Then, for sufficiently small, positive $t$, we have that

$$\| g \|_p^p - \| f \|_p < 0.$$

*Case* 4. *The order r is* 2 *and the index p is not an integer*

Define the polynomials $f$ and $g$ as in the previous case. We can proceed as before, but the analysis is more complicated in this case, because none of the products $\hat{h}(\gamma)\hat{k}(\gamma)$, for $\gamma$ in $\Gamma_1$, are $o(t^{2n})$ as $t \to 0$. Fortunately, there is an easier way.

Because $r = 2$, the characters $\gamma_j$ take the values $\pm$ 1 only. Therefore for small, positive $t$, both $f$ and $g$ are positive. Hence, using the branch of $z^p$ that is positive on the positive real axis, we have that

$$\| f \|_p^p = \int_G f(x)^p\, dx \quad \text{and} \quad \| g \|_p^p = \int_G g(x)^p\, dx.$$

It follows that everything depends on the behaviour as $t \to 0$ of the constant terms in the polynomials $f^p$ and $g^p$.

Expand $g^p$ by the binomial theorem. Again let $h$ be the sum of all the terms in the resulting double series that involve even powers of $v$, and let $k$ be the sum of all the terms that involve odd powers of $v$. Then

$$g^p = h + k \quad \text{and} \quad f^p = h - k.$$

The constant term in $h$ is $1 + O(t^2)$.

The first term in the series for $g^p$ that leads to a constant term in $k$ is the term $\binom{p}{n+1}(u+v)^{n+1}$; a constant term arises when $v$ is multiplied by $\prod_{j=1}^{n}(t\gamma_j)$. Thus, the constant term in $k$ is

$$\binom{p}{n+1}(n+1)!\, t^{2n} + O(t^{2n+2}).$$

Since $p$ is not an integer, we can choose $n$ so that $\binom{p}{n+1} < 0$. Then, for small, positive $t$, the constant term in $k$ is negative, while the constant term in $h$ is positive. For such $t$, the constant term in $f^p$ is greater than the constant term in $g^p$, and

$$\|f\|_p > \|g\|_p.$$

By the same analysis, this type of counterexample cannot arise if $r = 2$ and $p$ is an odd integer, for in this case the constant term in $k$ is positive for all small $t$.

*Case 5. The order r is 2 and the index p is an odd integer*

Let $n > p$ be an odd integer. For $-1 < t < 1$, consider the polynomial $f_t$ on $G$ given by

$$f_t(x) = \sum_{j=1}^{n} \gamma_j(x) + t \prod_{j=1}^{n} \gamma_j(x).$$

Note that, because $n$ is odd and $|t| < 1$, $f_t(x)$ is never 0. Hence, even if $p = 1$, the derivative $d/dt\,|f_t(x)|^p$ exists.

Let $F(t) = \|f_t\|_p^p$. We want $F'(0)$ to be negative, because if it is, then for small, positive $t$ we have that $\|f_t\|_p < \|f_{-t}\|_p$, although $f_t$ majorizes $f_{-t}$. Now

$$\begin{aligned} F'(0) &= \int_G \left[\frac{d}{dt}\,|f_t(x)|^p\right]_{t=0} dx \\ &= \int_G p\,|f_0(x)|^{p-1}[\operatorname{sgn} f_0(x)] \prod_{j=1}^{n} \gamma_j(x)\,dx. \end{aligned}$$

The integrand depends only on the number of $\gamma_j$ that are equal to $-1$ at $x$. Denoting this number by $s$, we have that

$$f_0(x) = n - 2s, \quad \text{and} \quad \prod_{j=1}^{n} \gamma_j(x) = (-1)^s.$$

Moreover, the set of all $x$ in $G$ for which exactly $s$ of the $\gamma_j$ are equal to $-1$ at $x$ has measure $2^{-n}\binom{n}{s}$. Finally, since $p-1$ is even, $|f_0|^{p-1} = f_0^{\,p-1}$. Thus

$$F'(0) = p2^{-n} \sum_{s=0}^{n} \binom{n}{s}(n-2s)^{p-1}\operatorname{sgn}(n-2s)(-1)^s.$$

Now the above sum is negative if $p$ is odd, $n > p$, and $n - p = 2 \bmod 4$. This fact is surely known, but there seems to be no reference for it; so we give a proof.

The sum is clearly 0 if $n$ is even. Suppose that $n$ is odd and $n \geqq p$. Let

$$\begin{aligned} H(\theta) &= \left(\frac{d}{d\theta}\right)^{p-1}(\sin\theta)^n \\ &= (2i)^{-n}\left(\frac{d}{d\theta}\right)^{p-1}(e^{i\theta} - e^{-i\theta})^n \\ &= (2i)^{-n} i^{p-1} \sum_{s=0}^{n}\binom{n}{s}(-1)^s(n-2s)^{p-1}e^{i(n-2s)\theta}. \end{aligned}$$

Let $\tilde{H}$ be the conjugate function of $H$. That is,

$$\tilde{H}(\theta) = (2i)^{-n} i^{p-1} \sum_{s=0}^{n}\binom{n}{s}(-1)^s(n-2s)^{p-1}[-i\operatorname{sgn}(n-2s)]e^{i(n-2s)\theta}.$$

Then

$$F'(0) = -p\, i^{n-p}\tilde{H}(0).$$

Now

$$\tilde{H}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\theta)\cot\left(\frac{-\theta}{2}\right)d\theta.$$

The conjugate-function integral is usually singular, but in this case, because $n > p - 1$, $H$ has a zero of order at least one at 0, and the integral converges absolutely. Since $p - 1$ is even, and $n$ is odd, $H$ is an odd function; thus

$$\tilde{H}(0) = -\frac{1}{\pi}\int_{0}^{\pi} H(\theta)\cot\left(\frac{\theta}{2}\right)d\theta.$$

Integrating by parts $p - 1$ times, we find that the integrated terms vanish at the end-points, and

$$\tilde{H}(0) = -\frac{1}{\pi}\int_{0}^{\pi}(\sin\theta)^n\left[\left(\frac{d}{d\theta}\right)^{p-1}\cot\left(\frac{\theta}{2}\right)\right]d\theta\,.$$

Now, since $p - 1$ is even, $(d/d\theta)^{p-1}\cot(\theta/2)$ is positive in the interval $(0, \pi)$. Hence $\tilde{H}(0) < 0$. To make $F'(0) < 0$ we need only arrange that $i^{n-p} = -1$. Since this happens if $n = p + 2$, the proof of the theorem is complete.

Remark. An inequality related to the one used above is that

$$\sum_{s=0}^{m+1}\binom{m+1}{s}(-1)^s(m+1-2s)^p\operatorname{sgn}(m+1-2s) < 0$$

if $p$ is odd, $m \geqq p$, and $m - p = 2 \bmod 4$. There are two equivalent ways to use this inequality to deal with Case 5. First, using the notation of Case 5 with $n = p + 2$, we can show that $\|f_{-1}\|_p > \|f_1\|_p$. Second, we can consider the polynomials $f$ and $g$ of Case 3, with $n = p + 1$ and $t = 1$, and show that $\|f\|_p > \|g\|_p$. The methods are equivalent because, under the stated conditions, the functions $|f_{-1}|$ and $|f|$ have the same distribution of values, as do $|f_1|$ and $|g|$.

### 3. Consequences of the main theorem

The exact majorant of a trigonometric polynomial $f$ is the polynomial $g$ satisfying $\hat{g} = |\hat{f}|$. In [2, p. 255], and in Cases 3, 4, and 5 above, various examples are given of polynomials $f$ with exact majorant $g$ such that

$$\|f\|_p > \|g\|_p;$$

in all cases, the coefficients of $f$ are real. With minor modifications, the arguments in this paper and in [1, p. 121] show that if $G$ is infinite and $p$ is finite and not an even integer, then for each constant $A > 0$ there exists a polynomial $f$, with real coefficients and exact majorant $g$, such that

$$\|f\|_p > A\|g\|_p.$$

We suppose for the rest of the paper that $1 < p < \infty$. [1, Th. 3, p. 126] lists 8 equivalent statements, the first being that $L^p(G)$ has the upper majorant property; hence all of these statements are false if $G$ is infinite and $p$ is not an even integer.

The second statement on list is that if $\phi$ is a function on $\Gamma$ such that $|\phi| \leqq \hat{g}$ for some function $g$ in $L^p(G)$, then $\phi \in L^p(G)^\wedge$, that is, there exists a function $f$ in $L^p(G)$ with $\hat{f} = \phi$. Thus, if $G$ is infinite and $p$ is not an even integer, there must exist a function $g$ in $L^p(G)$ and a function $\phi$ on $\Gamma$ such that $|\phi| \leqq \hat{g}$ but $\phi \notin L^p(G)^\wedge$. Write $\phi = \psi\hat{g}$ with $|\psi| \leqq 1$. Then $\psi$ is the average of two functions of absolute value 1. For one of these functions, $\psi_1$ say, we must have that $\psi_1\hat{g} \notin L^p(G)^\wedge$. In other words, if $G$ is infinite and $p$ is not an even integer, then there exists a function $c$ on $\Gamma$ such that $|c| \in L^p(G)^\wedge$ but $c \notin L^p(G)^\wedge$. This result is new for $p > 2$. For $p < 2$, it is proved by another method in [1, p. 122] and then used to show that $L^p(G)$ does not have the upper majorant property. By a different argument, we can show that the function $c$ above can be taken to be real-valued.

Statement 6 on Bachelis's list is the *lower majorant property for $L^q(G)$*, the dual space of $L^p(G)$. This asserts that for each function $f$ in $L^q(G)$ there exists a majorant $g$ in $L^q(G)$ with

$$\| g \|_q \leqq A(p) \| f \|_q.$$

Statement 7 is that each function in $L^q(G)$ has a majorant in $L^q(G)$. If $G$ is infinite, then, by the results of the present paper, these statements hold only if $p = \infty$ or $p$ is even, that is only if $q = 1$ or $q$ belongs to the sequence

$$2, \frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \cdots.$$

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